MODULE - 5

COMPLEX VARIABLE - RESIDUE INTEGRATION

LAURENT SERIES

It is the generalised form of Taylor series. Laurent series is the series expansion of $f(z)$ in powers of $z-z_0$ when $f(z)$ is not analytic at z_0. Taylor's series converges in a disk where as Laurent's series converges in an annulus with centre z_0.

Laurent theorem

Let $f(z)$ be analytic in a domain containing two concentric circle C_1 & C_2 with centre z_0 and the annulus between them. Let C be a simple closed path taken counter clockwise and encircles the inner circle. Then $f(z)$ can be represented by the Laurent series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Consisting of positive and negative powers.

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \& \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{1-n}} dz.$$

where $a_n (z-z_0)^n$ is the analytic part or Taylor part

and $\frac{b_n}{(z-z_0)^n}$ is the principal part

1. Find the Laurentz series expansion of $f(z) = \frac{e^z}{(z-1)^2}$ with centre at $z_0 = 1$

$$f(z) = \frac{1}{(z-1)^2} e^{z-1+1} = \frac{e \cdot e^{z-1}}{(z-1)^2}.$$

$$= e \cdot \frac{1}{(z-1)^2} \left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \cdots \right]$$

$$= e \left[\frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \cdots \right]$$

2. Find the Laurent series expansion of $f(z) = z^2 e^{1/z}$ with centre 0

$$f(z) = z^2 e^{1/z} = z^2 \left[\cancel{1+\frac{1}{z}} \; 1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \cdots \right.$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{z \cdot 3!} + \cdots$$

3. Find the Taylor series and Laurent series expansion of $f(z) = \frac{-2z+3}{z^2-3z+2}$

i) $|z| < 1$ ii) $1 < |z| < 2$ iii) $|z| > 2$

$$z^2-3z+2=0 \implies z=2,1$$

$$(z-2)(z-1)$$

$$\frac{-2z+3}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$-2z+3 = A(z-2) + B(z-1)$$

$$z=1 \implies A=-1$$

$$z=2 \implies B=-1$$

$$f(z) = \frac{-1}{z-1} + \frac{-1}{z-2}$$

i) $|z|<1$

$$f(z) = \frac{-1}{z-1} + \frac{-1}{z-2}$$

$$= \frac{-1}{-(1-z)} + \frac{-1}{-2(1-\frac{z}{2})}$$

$$= [1-z]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= 1+z+z^2+\cdots + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\right.$$

$|z|<1$

ii) $1<|z|<2$

$$1<|z| \implies \left|\frac{1}{z}\right|<1 \qquad |z|<2 \implies \left|\frac{z}{2}\right|<1$$

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= \left[\frac{-1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right] + \left[\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots\right]$$

$1 < |z| < 2.$

iii) $|z| > 2.$

$$2 < |z| \Rightarrow \frac{2}{|z|} < 1$$

$$|z| > 2 \Rightarrow |z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{z\left[1-\frac{1}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$|z| > 2$

2. Find the Taylor's and Laurent's series expansion of $f(z) = \dfrac{1}{z^2-4z+3}$ in the region

i) $|z|<1$ ii) $1<|z|<3$ iii) $|z|>3$

$$z^2-4z+3=0 \Rightarrow z=3,1$$

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$1 = A(z-3) + B(z-1)$$

put $z=3 \Rightarrow B = \frac{1}{2}$.

$z=1 \Rightarrow A = -\frac{1}{2}$.

$$f(z) = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{1}{2}}{z-3}$$

i) $|z|<1$ $\quad$ $|z|<1 \Rightarrow |z|<3$ $\quad$ z powers, $\left(\frac{z}{3}\right)$ powers

$\left|\frac{z}{3}\right|<1$

$$f(z) = \cancel{\frac{-1}{2(z-1)}} \frac{1}{2(1-z)} + \frac{1}{2(z-3)}$$

$$= \frac{1}{2(1-z)} + \frac{1}{2\times 3\left(\frac{z}{3}-1\right)}$$

$$= \frac{1}{2(1-z)} + \frac{-1}{6\left(1-\frac{z}{3}\right)}$$

$$= \frac{1}{2}(1-z)^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1+z+z^2+\cdots\right] - \frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right.$$

$|z|<1$

ii) $1<|z|<3$

$1<|z| \Rightarrow \left|\frac{1}{z}\right|<1$ $\qquad \left(\frac{1}{z}\right)$ power.

$|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$ $\qquad \left(\frac{z}{3}\right)$ powers.

$$f(z) = \frac{1}{2(1-z)} + \frac{1}{2(z-3)}$$

$$= \frac{1}{2\cdot z\left(\frac{1}{z}-1\right)} + \frac{1}{2\times 3\left(\frac{z}{3}-1\right)}$$

$$= \frac{-1}{2z\left(1-\frac{1}{z}\right)} - \frac{1}{6\left(1-\frac{z}{3}\right)}$$

$$= \frac{-1}{2z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}$$

$$= \frac{-1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right.$$

$$= \frac{-1}{2}\left[\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right.$$

$1<|z|<3$

iii) $|z|>3$

powers. powers

$\frac{3}{z}, \frac{1}{z}$

$|z|>3 \Rightarrow 3<|z|$

$\Rightarrow \left|\frac{3}{z}\right|<1$

$$f(z) = \frac{1}{2(1-z)} + \frac{1}{2(z-3)}$$

$|z|>3 \Rightarrow |z|>1$

$\Rightarrow 1<|z|$

$\Rightarrow \left|\frac{1}{z}\right|<1$

$$= \frac{1}{2z\left(\frac{1}{z}-1\right)} + \frac{1}{2z\left(1-\frac{3}{z}\right)}$$

$$= \frac{-1}{2z\left(1-\frac{1}{z}\right)} + \frac{1}{2z\left(1-\frac{3}{z}\right)}$$

$$= \frac{-1}{2z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1}$$

$$= \frac{-1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] + \frac{1}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\cdots\right.$$

$|z|>3$

3. Find the Laurent series expansion of

$$f(z) = \frac{z}{(z-1)(z-2)}$$

i) $0<|z-2|<1$

ii) $|z-1|>1$

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

$A = -1$

$B = 2.$

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}.$$

i) $0 < |z-2| < 1$

put $z-2 = u$. $\quad 0 < |u| < 1 \quad \frac{1}{u}$ powers $+ u$ power

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2} = \frac{-1}{u+1} + \frac{2}{u}.$$

$$f(z) = \frac{-1}{1+u} + \frac{2}{u} = \frac{2}{u} - \frac{1}{1+u} = \frac{2}{u} - (1+u)^{-1}$$

$$= \frac{2}{u} - \left[1 - u + u^2 - \cdots\right]$$

$$= \frac{2}{z-2} - \left[1 - (z-2) + (z-2)^2 - \cdots\right]$$

$0 < |z-2| < 1$

ii) $|z-1| > 1 \implies 1 < |z-1| \quad \left(\frac{1}{u}\right)$ powers.

put $z-1 = u$. $\quad 1 < |u| \implies \left|\frac{1}{u}\right| < 1$

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2} = \frac{-1}{u} + \frac{2}{u-1}$$

$$= \frac{-1}{u} + \frac{2}{u\left(1-\frac{1}{u}\right)} = \frac{-1}{u} + \frac{2}{u}\left[1 - \frac{1}{u}\right]^{-1}$$

$$= \frac{-1}{u} + \frac{2}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \cdots\right]$$

$$= \frac{-1}{u} + 2\left[\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots\right]$$

$$= \frac{-1}{z-1} + 2\left[\frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots\right] \quad |z-1| > 1$$

4. Find the Laurent's series expansion of

$$f(z) = \frac{1}{z - z^3} \quad ; \quad 1 < |z+1| < 2.$$

$$\frac{1}{z-z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1-z)(1+z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z}$$

put $z = 0 \implies 1 = A(1-z)(1+z) + Bz(1+z) + Cz(1-z)$

$1 = A$

put $z = 1 \implies B = 1/2$. put $z = -1 \implies C = -1/2$

$1 < |z+1| < 2$. $z+1 = u \implies z = u-1$

$$f(z) = \frac{1}{z} + \frac{1/2}{1-z} + \frac{-1/2}{1+z}.$$

$$= \frac{1}{u-1} + \frac{1/2}{2-u} - \frac{1/2}{u}.$$

$1 < |u| < 2$

$1 < |u|$

$\left|\frac{1}{u}\right| < 1$

$|u| < 2$

$\left|\frac{u}{2}\right| < 1$

$\left(\frac{1}{u}\right)^{\text{powers}} + \left(\frac{u}{2}\right)^{\text{powers}}$

$$f(z) = \frac{1}{+(u-1)} + \frac{1}{2(2-u)} - \frac{1}{2(u)}$$

$$= \frac{1}{+u\left[1-\frac{1}{u}\right]} + \frac{1}{2 \times 2\left(1-\frac{u}{2}\right)} - \frac{1}{2u}.$$

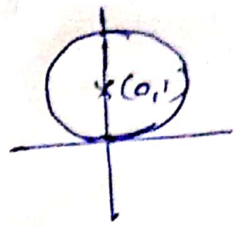
$$= \frac{+1}{u}\left[1-\frac{1}{u}\right]^{-1} + \frac{1}{4}\left[1-\frac{u}{2}\right]^{-1} - \frac{1}{2u}.$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right]+\frac{1}{4}\left[1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\cdots\right]-\frac{1}{2u}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\cdots\right]+\frac{1}{4}\left[1+\frac{z+1}{2}+\left(\frac{z+1}{2}\right)^2+\cdots\right]-\frac{1}{2(z+1)}$$

$1<|z+1|<2$

5. Find the Taylor's & Laurent's series expansion of $f(z)=\frac{1}{z^2}$ with centre at i



$z=0$ is the singular pt

centre at i i.e. $(0,1)$

$|z-i|<1$ & $|z-i|>1$

1) $|z-i|<1$ Let $z-i=u$. or $z=u+i$.

$|u|<1$ u power.

$$f(z)=\frac{1}{z^2}=\frac{1}{(u+i)^2}=\frac{1}{\left[i\left(\frac{u}{i}+1\right)\right]^2}$$

$$=\frac{-1}{\left[1+\frac{u}{i}\right]^2}=-\left[1+\frac{u}{i}\right]^{-2}$$

$$=-\left[1-2\frac{u}{i}+3\left(\frac{u}{i}\right)^2-\cdots\right]$$

$$=-1+2\left(\frac{z-i}{i}\right)+3(z-i)^2-\cdots$$

$|z-i|<1$

ii) $|z-i|>1$ Let $z-i=u$

$\Longrightarrow |u|>1 \Longrightarrow \left|\frac{1}{u}\right|<1$

$$f(z) = \frac{1}{(u+i)^2} = \frac{1}{\left[u\left(1+\frac{i}{u}\right)\right]^2}$$

$$= \frac{1}{u^2} \cdot \frac{1}{\left(1+\frac{i}{u}\right)^2} = \frac{1}{u^2}\left[1+\frac{i}{u}\right]^{-2}$$

$$= \frac{1}{u^2}\left[1 - \frac{2i}{u} + 3\left(\frac{i}{u}\right)^2 - 4\left(\frac{i}{u}\right)^3 + \cdots\right.$$

$$= \frac{1}{(z-i)^2}\left[1 - \frac{2i}{z-i} + \frac{3i^2}{(z-i)^2} - \cdots\right.$$

$$= \frac{1}{(z-i)^2}\left[1 - \frac{2i}{z-i} - \frac{3}{(z-i)^2} + \cdots\right.$$

$|z-i|>1$

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Zeros of an Analytic functions

A point $z = z_0$ in a domain D in which $f(z_0) = 0$ is called a zero of an analytic function. A zero has order n if not only f but also the derivatives $f', f'', \dots f^{n-1}$ are all zero at $z = z_0$ & $f^n(z_0) \neq 0$.

SINGULARITIES OF AN ANALYTIC FUNCTION

A function $f(z)$ has a singular point or singularity at a point $z = z_0$ if $f(z)$ is not analytic at $z = z_0$. The singular point of $z = z_0$ is called an isolated singular point of $f(z)$ if there is no other singular point of $f(z)$ in a small neighbourhood of z_0. Otherwise it is called a non-isolated singular point.

Eg: $f(z) = \tan z$. has singular points

$z = \pm \pi/2, \pm 3\pi/2 \dots$ Each of these points is an isolated singular point.

CLASSIFICATION OF SINGULARITIES

1. POLE

If the principal part of the Laurent series has only finitely many terms & it is of the form $\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$

Then the singularity of $f(z)$ at $z = z_0$ is called a pole of order m is called its order.

Poles of the first order are known as simple poles

Eg: $f(z) = \frac{\sin z}{z^5}$

$z = 0$ is the singular point

$$f(z) = \frac{\sin z}{z^5} = \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^3 \cdot 3!} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots\cdots$$

$z = 0$ is a pole of order 4

ESSENTIAL SINGULARITY

If the principal part of the Laurent series expansion of the function $f(z)$ has infinite number of terms then the point $z = z_0$ is called an essential singular point

$f(z) = e^{1/z}$ $\quad z = 0$ is the singular point

$$= 1 + \frac{1}{z \cdot 1!} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + \cdots\cdots$$

$z = 0$ is an essential singularity

REMOVABLE SINGULARITY

If the principal part of the Laurent series expansion of the function $f(z)$ have no terms then the point $z = z_0$ is called a removable singular point

Eg: $f(z) = \frac{\sin z}{z}$ $z=0$ is the singular point

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\cdots$$

$z=0$ is a removable singularity

I. Describe the nature & type of singularities in the following functions.

1. $f(z) = z \cdot \sin\left(\frac{1}{z}\right)$

$z=0$ is the singular point

$$z \cdot \sin\left(\frac{1}{z}\right) = \cancel{\$}\, z \cdot \left[\frac{1}{z} - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} - \frac{1}{z^7 \cdot 7!} + \cdots\right.$$

$$= 1 - \frac{1}{z^2 \cdot 3!} + \frac{1}{z^4 \cdot 5!} - \frac{1}{z^6 \cdot 7!} + \cdots\cdots$$

$z=0$ is an essential singularity

2. $\frac{e^{-z^2}}{z^2}$ $z=0$ is the singular point

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\cdots$$

$$e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\cdots$$

$$\frac{e^{-z^2}}{z^2} = \frac{1}{z^2}\left[1 - \frac{z^2}{1!} + \frac{z^4}{2!} - \frac{z^6}{3!} + \cdots\right.$$

$$= \frac{1}{z^2} - \frac{1}{1!} + \frac{z^2}{2!} - \frac{z^4}{3!} + \cdots$$

$z = 0$ is a pole of order 2

3) $z^2 \cos\left(\frac{1}{z}\right)$

$z = 0$ is the singular point

$$f(z) = z^2\left[1 - \frac{1}{z^2 \cdot 2!} + \frac{1}{z^4 \cdot 4!} - \frac{1}{z^6 \cdot 6!} + \cdots\right.$$

$$= z^2 - \frac{1}{2!} + \frac{1}{z^2 \cdot 4!} - \frac{1}{z^4 \cdot 6!} + \cdots$$

$z = 0$ is an essential singularity

4) $(z+i)^2 e^{\frac{1}{z+i}}$

$z = -i$ is the singular point

$$f(z) = (z+i)^2\left[1 + \frac{1}{(z+i)!} + \frac{1}{(z+i)^2 \cdot 2!} + \frac{1}{(z+i)^3 \cdot 3!} + \cdots\right.$$

$$= (z+i)^2 + \frac{z+i}{1!} + \frac{1}{2!} + \frac{1}{(z+i) \cdot 3!} + \cdots$$

$z = -i$ is an essential singularity

5. $f(z) = \dfrac{\sin z}{(z-\pi)^2}$

$z=\pi$ is the singular point

$$f(z) = \frac{\sin z}{(z-\pi)^2} = \frac{(\sin(z-\pi+\pi)}{(z-\pi)^2} = \frac{-\sin(z-\pi)}{(z-\pi)^2}$$

$$= \frac{-1}{(z-\pi)^2}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\right.$$

$$= -\left[\frac{1}{z-\pi} - \frac{z-\pi}{3!} + \frac{(z-\pi)^3}{5!} - \cdots\right.$$

$z=\pi$ is a pole of order 1

6. $f(z) = \dfrac{z - \sin z}{z^2}$ $\quad z=0$ is the singulae point

$$= \frac{1}{z^2}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\right)\right]$$

$$= \frac{1}{z^2}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right.$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots$$

$z=0$ is a removable singularity

Note: $\lim\limits_{z\to a} f(z) = \infty \quad z=a$ is a pole.

$\lim\limits_{z\to a} f(z)$ is not defined $\rightarrow$ $z=a$ is an essential singu.

$\lim\limits_{z\to a} f(z)$ exist finitely $\rightarrow$ $z=a$ is a removable singu

7. $f(z) = \dfrac{e^z}{z+\sin z}$. $z=0$ is the singular point

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{e^z}{z+\sin z} = \dfrac{1}{0} = \infty$.

$z=0$ is a pole.

$g(z) = z+\sin z$. $\quad g(0)=0$

$g'(z) = 1+\cos z \quad g'(0) = 2$.

$g'(0) \neq 0$.

$z=0$ is a pole of order 1

RESIDUE INTEGRATION

RESIDUE :- The coefficient of $\dfrac{1}{z-z_0}$ in the Laurent series expansion of $f(z)$ is called the residue of $f(z)$ at $z=z_0$ $\left[\operatorname{Res}_{z=z_0} f(z)\right]$

RESIDUE AT SIMPLE POLE (pole of order 1)

Let $z=z_0$ be a simple pole of $f(z)$ then residue at $z=z_0$ is

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0)\,f(z)$$

If $f(z) = \dfrac{P(z)}{q(z)}$ then $\operatorname{Res}_{z=z_0} f(z) = \dfrac{P(z_0)}{q'(z_0)}$ where $P(z_0)\neq 0$

RESIDUE AT MULTIPLE POLES [Pole of order greater than 1]

If $z = z_0$ is a pole of order m then residue of $f(z)$ at $z = z_0$ is

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \operatorname*{Lt}_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) \right]$$

1. Find the residues of $f(z) = \frac{1}{(z-1)(z-2)^2}$ at $z = 1, 2$.

$z = 1$ is a pole of order 1

$z = 2$ is a pole of order 2

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=1} f(z) = \operatorname*{Lt}_{z \to 1} (z-1) \frac{1}{(z-1)(z-2)^2}$$

$$= \operatorname*{Lt}_{z \to 1} \frac{1}{(z-2)^2} = 1$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{(2-1)!} \operatorname*{Lt}_{z \to 2} \frac{d^{2-1}}{dz^{2-1}} (z-2)^2 \cdot \frac{1}{(z-1)(z-2)^2}$$

$$= \operatorname*{Lt}_{z \to 2} \frac{d}{dz} \left(\frac{1}{(z-1)} \right)$$

$$= \operatorname*{Lt}_{z \to 2} \frac{-1}{(z-1)^2} = \underline{\underline{-1}}$$

RESIDUE THEOREM

Let $f(z)$ be analytic inside a simple closed path C & on C except for finitely many singular points. $z_1, z_2 \ldots z_k$ inside C. then.

$$\int_C f(z)\,dz = 2\pi i \left[\text{Sum of Residues} \right]$$

taken counter clockwise.

1. Evaluate $\int_C \frac{1}{(z-1)(z-2)^2} dz$ where $C: |z-2| = \frac{1}{2}$

$z = 1, 2$ are the singular points

$z=1$ is a pole of order 1 which lies outside C.

$z = 2$ is a pole of order 2 which lies inside C.

$R_1 = \underset{z=1}{\text{Res}}\, f(z) = 0$

$R_2 = \underset{z=2}{\text{Res}}\, f(z) = -1$

$$\int_C \frac{1}{(z-1)(z-2)^2} dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i [0-1]$$

$$= \underline{\underline{-2\pi i}}$$

2. $\int_C \frac{5z+7}{z^2+2z-3} dz \qquad C: |z-2| = 2$

$z = 1, -3$ are the singular points

$z=1$ is a pole of order 1 which lies inside C

$z=-3$ is a pole of order 1 which lies outside C

$$\underset{z=1}{\text{Res}}\, f(z) = \underset{z\to 1}{Lt}\ (z-1) \cdot \frac{5z+7}{(z-1)(z+3)}$$

$$= \underline{\underline{3}}$$

$$\int_C \frac{5z+7}{z^2+2z-3} dz = 2\pi i \times 3 = \underline{\underline{6\pi i}}$$

3) $\int_C \frac{1-z}{z^3(1+z)} dz \qquad C: |z| = \frac{1}{2}$

$z = 0, -1$ are the singular points.

$z = 0$ is a pole of order 3 which lies inside C

$z = -1$ is a pole of order 1 which lies outside C

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{2!} \underset{z\to 0}{Lt} \frac{d^2}{dz^2} (z-0)^3 \cdot \frac{1-z}{z^3(1+z)}$$

$$= \frac{1}{2} \cdot \underset{z\to 0}{Lt} \frac{d^2}{dz^2} \frac{1-z}{1+z}.$$

$$= \frac{1}{2} \underset{z\to 0}{Lt} \frac{d}{dz} \frac{(1+z)\times -1 - (1-z) \cdot 1}{(1+z)^2}.$$

$$= \frac{1}{2} \underset{z\to 0}{Lt} \frac{d}{dz} \frac{-2}{(1+z)^2}.$$

$$= \underset{z\to 0}{Lt} \frac{+2}{(1+z)^3} = \underline{\underline{2}}$$

$$\int_C \frac{1-z}{z^3(1+z)} dz = 2\pi i \times 2 = \underline{\underline{4\pi i}}$$

4. Using Cauchy's residue theorem. Evaluate.

$\int_C \frac{z^2}{(z-1)^2 (z+2)} dz \qquad C: |z| = 3$

$z = 1, -2$ are the singular points

$z = 1$ is a pole of order 2 which lies inside C

$z = -2$ is a pole of order 1 which lies inside C

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!} \operatorname*{Lt}_{z\to 1} \frac{d}{dz} (z-1)^2 \cdot \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \operatorname*{Lt}_{z\to 1} \frac{d}{dz} \frac{z^2}{z+2}$$

$$= \operatorname*{Lt}_{z\to 1} \frac{(z+2)\cdot 2z - z^2}{(z+2)^2} = \frac{5}{9}$$

$$\operatorname*{Res}_{z=-2} f(z) = \operatorname*{Lt}_{z\to -2} (z+2) \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \operatorname*{Lt}_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$$\int_C \frac{z^2}{(z-1)^2 (z+2)} dz = 2\pi i \left[\frac{5}{9} + \frac{4}{9}\right] = 2\pi i$$

5. $\displaystyle\int_C \frac{z-23}{z^2-4z-5} dz \qquad c: |z-2-i| = 3.5$

$z = -1, 5$ are the singular points

$z = -1$ is a pole of order 1 which lies inside C

$z = 5$ is a pole of order 1 which lies inside C

$$\operatorname*{Res}_{z=-1} f(z) = \operatorname*{Lt}_{z\to -1} (z+1) \cdot \frac{z-23}{(z+1)(z-5)}$$

$$= \frac{24}{6} = 4$$

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$$\operatorname*{Res}_{z=5} f(z) = \lim_{z \to 5} (z-5) \cdot \frac{z-23}{(z+1)(z-5)} = -3$$

$$\int_C \frac{z-23}{z^2-4z-5}\, dz = 2\pi i \left[4-3\right] = \underline{\underline{2\pi i}}$$

6. Evaluate $\displaystyle\int_C \frac{1}{z^2+4iz-1}\, dz \qquad c: |z|=1$

$$z^2+4iz-1 = 0 \implies z = -2i+\sqrt{3}i,\ -2i-\sqrt{3}i$$

$$z_1 = -2i+\sqrt{3}i \qquad z_2 = -2i-\sqrt{3}i$$

$z_1 = -2i+\sqrt{3}i$ is a pole of order 1 which lies inside C.

$z_2 = -2i-\sqrt{3}i$ is a pole of order 1 which lies outside C

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} (z-z_1) \cdot \frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z_1-z_2} = \frac{1}{2\sqrt{3}i}$$

$$\therefore \int_C \frac{1}{z^2+4iz-1}\, dz = 2\pi i \times \frac{1}{2\sqrt{3}i} = \underline{\underline{\frac{\pi}{\sqrt{3}}}}$$

7. $\displaystyle\int_C \frac{dz}{(z^2+4)^2}\, dz \qquad c: |z-i|=2$

$z = \pm 2i$ are the singular points

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$Z = 2i$ is a pole of order 2 which lies inside C

$Z = -2i$,, outside C

$$\int_C \frac{1}{(z+2i)^2 (z-2i)^2} dz.$$

$$\underset{z=2i}{\text{Res}} f(z) = \frac{1}{1!} \underset{z \to 2i}{Lt} \frac{d}{dz} (z-2i)^2 \frac{1}{(z-2i)^2 (z+2i)^2}$$

$$= \underset{z \to 2i}{Lt} \frac{d}{dz} \frac{1}{(z+2i)^2}$$

$$= \underset{z \to 2i}{Lt} \frac{-2}{(z+2i)^3} = \frac{-2}{(4i)^3} = \frac{1}{32i}$$

$$\int_C \frac{1}{(z^2+4)^2} dz = 2\pi i \times \frac{1}{32i} = \underline{\underline{\frac{\pi}{16}}}.$$

8. Using Cauchy's residue theorem

Evaluate $\int_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)} dz \quad c: |z| = 1$

$$\int_C \frac{30z^2 - 23z + 5}{4\left(z-\frac{1}{2}\right)^2 3\left(z-\frac{1}{3}\right)} = \frac{1}{12}\int_C \frac{30z^2-23z+5}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{3}\right)} dz$$

$z = \frac{1}{2}, \frac{1}{3}$ are the singular pts

$z = \frac{1}{2}$ is a pole of order 2 which lies inside C.

$z = \frac{1}{3}$ is a pole of order 1 which lies inside C.

$$\operatorname*{Res}_{z=\frac{1}{2}} f(z) = \frac{1}{1!} \operatorname*{Lt}_{z\to\frac{1}{2}} \frac{d}{dz} \left(z-\frac{1}{2}\right)^2 \cdot \frac{30z^2-23z+5}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{3}\right)}$$

$$= \operatorname*{Lt}_{z\to\frac{1}{2}} \frac{d}{dz} \frac{30z^2-23z+5}{z-\frac{1}{3}}$$

$$= \operatorname*{Lt}_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{3}\right)\cdot(60z-23) - (30z^2-23z+5)}{\left(z-\frac{1}{3}\right)^2}$$

$$= \frac{1}{2}.$$

$$\operatorname*{Res}_{z=\frac{1}{3}} f(z) = \operatorname*{Lt}_{z\to\frac{1}{3}} \left(z-\frac{1}{3}\right) \frac{30z^2-23z+5}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2}.$$

$$= 2$$

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{2} + 2\right] = \underline{\underline{5\pi i}}$$

9) Evaluate $\int_C \frac{e^z}{\cos \pi z} dz$ where C is the unit circle $|z|=1$ using Residue theorem

$\cos \pi z = 0 \implies \pi z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$z = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots\ldots$ are the singular points

$z = \pm\frac{1}{2}$ are simple poles (order 1) which lies inside C.

$$\frac{e^z}{\cos \pi z} = \frac{P(z)}{q(z)}$$

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{P(z_0)}{q'(z_0)}$$

$$= \frac{e^{1/2}}{-\pi \sin \frac{\pi}{2}} = \frac{-e^{1/2}}{\pi}$$

$$\operatorname*{Res}_{z=-1/2} f(z) = \frac{e^{-1/2}}{\pi \sin \frac{\pi}{2}} = \frac{e^{-1/2}}{\pi}$$

$$\int_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left[-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right]$$

$$= 2i \left[e^{-1/2} - e^{1/2} \right]$$

EVALUATION OF REAL INTEGRALS

CONTOUR INTEGRATION

Integrals of rational functions of $\sin\theta$, $\cos\theta$.

Consider an Integral of the form

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

Setting $z = e^{i\theta}$ we have.

$$\boxed{\begin{aligned} \cos\theta &= \frac{z^2+1}{2z}, \\ \sin\theta &= \frac{z^2-1}{2iz}. \\ d\theta &= \frac{dz}{iz}. \end{aligned}}$$

As θ varies from 0 to 2π, the variable z transverses once around the unit circle $|z|=1$ in the anticlockwise direction. Then the Integral can be written as

$$\oint_C f(z)\, \frac{dz}{iz} \qquad C : |z| = 1$$

This contour Integral can be evaluated by using residue theorem.

1. Evaluate $\int_0^{2\pi} \frac{1}{2+\sin\theta}\, d\theta$

put $z = e^{i\theta}$ $\quad \sin\theta = \frac{z^2-1}{2iz}$ $\quad d\theta = \frac{dz}{iz}$.

$$\int_{C:|z|=1} \frac{1}{2+\frac{z^2-1}{2iz}} \frac{dz}{iz}$$

$$= \int_C \frac{1 \times 2iz}{(4iz+z^2-1)} \frac{dz}{iz}$$

$$= 2\int_C \frac{dz}{z^2+4iz-1}$$

$z^2+4iz-1 = 0 \implies z = -2i \pm \sqrt{3}i$

Let $z_1 = -2i+\sqrt{3}i$ $\quad z_2 = -2i-\sqrt{3}i$

z_1 is a pole of order 1 which lies inside C & z_2 is a pole of order 1 which lies outside C.

$$\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Lt}_{z\to z_1} (z-z_1)\cdot\frac{1}{(z-z_1)(z-z_2)}$$

$$= \operatorname*{Lt}_{z\to z_1} \frac{1}{z-z_2} = \frac{1}{z_1-z_2}$$

$$= \frac{1}{2\sqrt{3}\,i}$$

$$\therefore \int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = 2\cdot\int_C \frac{dz}{z^2+4iz-1}$$

$$= 2\cdot 2\pi i \times \frac{1}{2\sqrt{3}\,i}$$

$$= \underline{\underline{\frac{2\pi}{\sqrt{3}}}}$$

2. $$\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\,d\theta$$

$$\cos\theta = \frac{z^2+1}{2z} \qquad d\theta = \frac{dz}{iz} \qquad C:|z|=1$$

$$\int_C \frac{1}{\sqrt{2}-\left(\frac{z^2+1}{2z}\right)}\,\frac{dz}{iz}$$

$$= \int_C \frac{1 \times 2z}{2\sqrt{2}z-z^2-1}\,\frac{dz}{iz}$$

$$= \frac{-2}{i}\int_C \frac{1}{z^2-2\sqrt{2}z+1}\,dz$$

$$z^2-2\sqrt{2}z+1=0 \implies \sqrt{2}\pm 1$$

$$z_1 = \sqrt{2}+1 \qquad z_2 = \sqrt{2}-1$$

$z_1 = \sqrt{2}+1$ is a pole of order 1 which lies, outside C and $z_2 = \sqrt{2}-1$ is a pole of order 1 which lies inside C

$$\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta = \frac{-2}{i} \int_C \frac{1}{(z-z_1)(z-z_2)}\, dz$$

$$\operatorname{Res}_{z=z_2} f(z) = \operatorname{Lt}_{z\to z_2} (z-z_2) \frac{1}{(z-z_1)(z-z_2)}\, dz$$

$$= \frac{1}{z_2 - z_1} = \frac{1}{-2}$$

$$\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta = \frac{-2}{i} \times 2\pi i \times \frac{1}{-2} = \underline{\underline{2\pi}}$$

3. $$\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \int_0^{2\pi} \frac{\operatorname{Im.p}(e^{i\theta})}{3+\cos\theta}\, d\theta$$

$\operatorname{Im\,part}(e^{i\theta}) = \sin\theta$

$\operatorname{R.part}(e^{i\theta}) = \cos\theta$

$$= \operatorname{Im.part} \int_C \frac{z}{3+\frac{z^2+1}{2z}} \frac{dz}{iz}$$

$z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$C : |z| = 1$

$$= \operatorname{Im.p.} \int_C \frac{z \times 2z}{6z+z^2+1} \frac{dz}{iz}$$

$$= \text{Im.p}\ \frac{2}{i} \int_{C} \frac{z}{z^2+6z+1}\, dz$$

$$z^2+6z+1 = 0 \implies -3 \pm 2\sqrt{2}$$

$z_1 = -3+2\sqrt{2}$ is a pole of order 1 which lies inside C. $z_2 = -3-2\sqrt{2}$ is a pole of order 1 which lies outside C

$$\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta = \text{Im.p of } \frac{2}{i} \int_C \frac{z}{(z-z_1)(z-z_2)}\, dz$$

$$\underset{z=z_1}{\text{Res}} f(z) = \lim_{z \to z_1} (z-z_1) \cdot \frac{z}{(z-z_1)(z-z_2)}$$

$$= \frac{z_1}{z_1-z_2} = \frac{-3+2\sqrt{2}}{4\sqrt{2}}.$$

$$\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta = \text{Im. p of } \frac{2}{i} \cdot 2\pi i \left(\frac{-3+2\sqrt{2}}{4\sqrt{2}}\right)$$

$$= \text{Im.p of } 4\pi \left(\frac{-3+2\sqrt{2}}{4\sqrt{2}}\right)$$

$$= \underline{\underline{0}}.$$

4. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+4\cos 2\theta}\, d\theta$ $\qquad z = e^{i\theta}$

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2}$$

$$z = e^{i\theta} \qquad \cos 2\theta = \frac{z^4+1}{2z^2} \qquad = \frac{z^2 + \frac{1}{z^2}}{2}$$

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$$= \int_0^{2\pi} \frac{R.p \text{ of } (e^{i\theta})}{5 + 4\cos 2\theta} \, d\theta$$

$$= \int_C \frac{R.p \text{ of } (z)}{5 + 4\left(\frac{z^4+1}{2z^2}\right)} \frac{dz}{iz}$$

$$= R.p \text{ of } \int_C \frac{z \times 2z^2}{10z^2 + 4z^4 + 4} \cdot \frac{dz}{iz}$$

$$= R.p \ \frac{2}{i} \int_C \frac{z^2}{2\left[2z^4 + 5z^2 + 2\right]} dz$$

$$= R.p \ \frac{1}{2i} \int_C \frac{z^2}{\left[z^4 + \frac{5}{2}z^2 + 1\right]} dz$$

Let $z^2 = u$

$$z^4 + \frac{5}{2}z^2 + 1 \Rightarrow u^2 + \frac{5}{2}u + 1 = 0$$

$$\Rightarrow u = \frac{-1}{2} \text{ or } -2$$

$$\Rightarrow z^2 = \frac{-1}{2} \Rightarrow z = \pm \frac{i}{\sqrt{2}}$$

$$z^2 = -4 \Rightarrow z = \pm\sqrt{2}\, i$$

$z = \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}$ are simple poles which lies inside C and $z = \pm\sqrt{2}\,i$ lies outside C

$$\operatorname*{Res}_{z=\frac{i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{i}{\sqrt{2}}} \left(z-\frac{i}{\sqrt{2}}\right) \cdot \frac{z^2}{\left(z-\frac{i}{\sqrt{2}}\right)\left(z+\frac{i}{\sqrt{2}}\right)\left(z-\sqrt{2}i\right)\left(z+\sqrt{2}i\right)}$$

$$= \frac{-1/2}{\sqrt{2}i \times \frac{3}{2}} = \frac{-1}{3\sqrt{2}i}$$

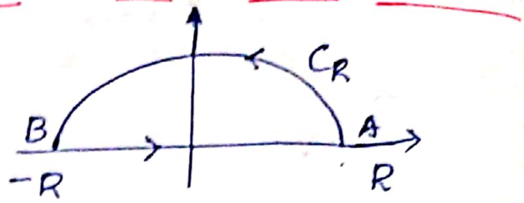
$$\operatorname*{Res}_{z=\frac{-i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{-i}{\sqrt{2}}} \left(z+\frac{i}{\sqrt{2}}\right) \frac{z^2}{\left(z-\frac{i}{\sqrt{2}}\right)\left(z+\frac{i}{\sqrt{2}}\right)\left(z-\sqrt{2}i\right)\left(z+\sqrt{2}i\right)}$$

$$= \frac{-1/2}{-\sqrt{2}i \times \frac{3}{2}} = \frac{1}{3\sqrt{2}i}$$

$$\int_0^{2\pi} \frac{\cos\theta}{3+4\cos 2\theta}\, d\theta = R.p \text{ of } \frac{1}{2i} \times 2\pi i \left[\frac{-1}{3\sqrt{2}i} + \frac{1}{3\sqrt{2}i}\right]$$

$$= \cancel{Rp}\ \underline{\underline{0}}.$$

IMPROPER REAL INTEGRALS OF THE FORM $\int_{-\infty}^{\infty} f(x)\,dx$.



Consider the contour integral $\oint_C f(z)\,dz$ around a path C from A to B along the Semicircle C_R and then from B to A along the real axis.

By the residue theorem

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx.$$

$$= 2\pi i \left[\begin{array}{l}\text{Sum of the residues} \\ \text{in the upper half}\end{array}\right]$$

or

above the real axis.

1. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\,dx$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\,dx = \int_C \frac{1}{(1+z^2)^3}\,dz.$$

$1+z^2=0 \implies z = \pm i$ are the singular points

$z = i$ is a pole of order 3 which lies inside C and $z=-i$ is a pole of order 3 which lies outside C.

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2!} \underset{z\to i}{Lt} \frac{d^2}{dz^2} (z-i)^3 \cdot \frac{1}{(z-i)^3 (z+i)^3}$$

$$= \frac{1}{2} \cdot \underset{z\to i}{Lt} \frac{d^2}{dz^2} \frac{1}{(z+i)^3}$$

$$= \frac{1}{2} \cdot \underset{z\to i}{Lt} \frac{d}{dz} \frac{-3}{(z+i)^4}$$

$$= \frac{-3}{2} \underset{z\to i}{Lt} \frac{-4}{(z+i)^5} = 6 \cdot \frac{1}{(2i)^5}$$

$$= \frac{6}{32i} = \frac{3}{16i}$$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx = 2\pi i \times \frac{3}{16i} = \underline{\underline{\frac{3\pi}{8}}}$$

2. $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^2} dx$. Evaluate this integral

$$\int_0^{\infty} \frac{1}{(1+z^2)^2} dz = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(1+z^2)^2} dz.$$

$z = \pm i$ are the singular points

$z = i$ is a pole of order 2 which lies inside C and $z = -i$ is a pole of order 2. which lies outside C

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{1!} \underset{z\to i}{Lt} \frac{d}{dz} (z-i)^2 \cdot \frac{1}{(z-i)^2 (z+i)^2}$$

$$= \underset{z\to i}{Lt} \frac{d}{dz} \frac{1}{(z+i)^2} =$$

$$= \underset{z\to i}{Lt} \frac{-2}{(z+i)^3} = \frac{-2}{8 \times -i} = \frac{1}{4i}$$

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\times 2\pi i\times\frac{1}{4i} = \frac{\pi}{4}$$

3. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$. using contour integration.

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \int_C \frac{1}{z^4+1}\,dz$$

Singular points $= e^{i(2k+1)\frac{\pi}{4}}$

$k=0,1,2,3$

$$z^4+1 = 0$$

$$z_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}.$$

$$z_2 = e^{i3\pi/4} = \frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}.$$

$$z_3 = e^{i\frac{5\pi}{4}} = \frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}.$$

$$z_4 = e^{i7\pi/4} = \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

z_1, z_2 are poles of order 1 which lies inside C (ie in the upper half)

z_3, z_4 are poles of order 1 which lies outside C (ie in the lower half)

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{P(z_1)}{q'(z_1)} = \frac{1}{4z_1^3} = \frac{z_1}{4z_1^4}$$

$$= -\frac{(1+i)}{4\sqrt{2}}$$

we have $z_1^4 = -1$

$z_2^4 = -1$

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{P(z_2)}{q'(z_2)} = \frac{1}{4z_2^3} = \frac{z_2}{4z_2^4}$$

$$-\frac{(-1+i)}{4\sqrt{2}}$$

$$\int_C f(z)\,dz = 2\pi i \left[-\frac{(1+i)}{4\sqrt{2}} - \frac{(-1+i)}{4\sqrt{2}}\right]$$

$$= \underline{\underline{\frac{\pi}{\sqrt{2}}}}.$$